Chapter 7: Link Analysis-2

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Road map

- PageRank
- HITS
- Summary

PageRank

- The year 1998 was an eventful year for Web link analysis models. Both the PageRank and HITS algorithms were reported in that year.
- The connections between PageRank and HITS are quite striking.
- Since that eventful year, PageRank has emerged as the dominant link analysis model,
  - due to its query-independence,
  - its ability to combat spamming, and
  - Google’s huge business success.

PageRank: the intuitive idea

- PageRank relies on the democratic nature of the Web by using its vast link structure as an indicator of an individual page's value or quality.
- PageRank interprets a hyperlink from page *x* to page *y* as a vote, by page *x*, for page *y*.
- However, PageRank looks at more than the sheer number of votes; it also analyzes the page that casts the vote.
    - Votes casted by “important” pages weigh more heavily and help to make other pages more "important."
- This is exactly the idea of **rank prestige** in social network.

More specifically

- A hyperlink from a page to another page is an implicit conveyance of authority to the target page.
  - The more in-links that a page *i* receives, the more prestige the page *i* has.
- Pages that point to page *i* also have their own prestige scores.
  - A page of a higher prestige pointing to *i* is more important than a page of a lower prestige pointing to *i*.
  - In other words, a page is important if it is pointed to by other important pages.

PageRank algorithm

- According to rank prestige, the importance of page *i* (*i*'s PageRank score) is the sum of the PageRank scores of all pages that point to *i*.
- Since a page may point to many other pages, its prestige score should be shared.
- The Web as a directed graph *G* = (*V*, *E*). Let the total number of pages be *n*. The PageRank score of the page *i* (denoted by *P*(*i*)) is defined by:

$$P(i) = \sum_{(j,i)\in E} \frac{P(j)}{O_j},$$

O_j is the number of out-link of j

Matrix notation

- We have a system of n linear equations with n unknowns. We can use a matrix to represent them.
- Let $\boldsymbol{P}$ be a n-dimensional column vector of PageRank values, i.e., $\boldsymbol{P} = (P(1), P(2), \ldots, P(n))^T$.
- Let $\boldsymbol{A}$ be the adjacency matrix of our graph with

$$A_{ij} = \begin{cases} \dfrac{1}{O_i} & if\ (i, j) \in E \\ 0 & otherwise \end{cases} \qquad (14)$$

- We can write the n equations with (PageRank)

$$\boldsymbol{P} = \boldsymbol{A}^T \boldsymbol{P} \qquad (15)$$

Solve the PageRank equation

$$\boldsymbol{P} = \boldsymbol{A}^T \boldsymbol{P} \qquad (15)$$

- This is the characteristic equation of the **eigensystem**, where the solution to ***P*** is an **eigenvector** with the corresponding **eigenvalue** of 1.
- It turns out that if ***some conditions*** are satisfied, 1 is the largest **eigenvalue** and the PageRank vector ***P*** is the **principal eigenvector**.
- A well known mathematical technique called **power iteration** can be used to find ***P***.
- **Problem**: the above Equation does not quite suffice because the Web graph does not meet the conditions.

Using Markov chain

- To introduce these conditions and the enhanced equation, let us derive the same Equation (15) based on the **Markov chain**.
  - In the Markov chain, each Web page or node in the Web graph is regarded as a state.
  - A hyperlink is a transition, which leads from one state to another state with a probability.
- This framework models Web surfing as a stochastic process.
- It models a Web surfer randomly surfing the Web as state transition.

Random surfing

- Recall we use O_i to denote the number of out-links of a node *i*.
- Each transition probability is $1/O_i$ if we assume the Web surfer will click the hyperlinks in the page *i* uniformly at random.
  - The “back” button on the browser is not used and
  - the surfer does not type in an URL.

Transition probability matrix

- Let ***A*** be the state transition probability matrix,,

$$
\boldsymbol{A} = \begin{pmatrix}
A_{11} & A_{12} & . & . & . & A_{1n} \\
A_{21} & A_{22} & . & . & . & A_{2n} \\
. & . & & & & . \\
. & . & & & & . \\
. & . & & & & . \\
A_{n1} & A_{n2} & . & . & . & A_{nn}
\end{pmatrix}
$$

- A_{ij} represents the transition probability that the surfer in state *i* (page *i*) will move to state *j* (page *j*). A_{ij} is defined exactly as in Equation (14).

$$
A_{ij} = \begin{cases} \dfrac{1}{O_i} & if\,(i, j) \in E \\ 0 & otherwise \end{cases}
$$

Let us start

- Given an initial probability distribution vector that a surfer is at each state (or page)
  - $\boldsymbol{p}_0 = (p_0(1), p_0(2), \ldots, p_0(n))^T$ (a column vector) and
  - an $n \times n$ **transition probability matrix** $\boldsymbol{A}$,

we have

$$\sum_{i=1}^{n} p_0(i) = 1 \qquad (16)$$

$$\sum_{j=1}^{n} A_{ij} = 1 \qquad (17)$$

- If the matrix $\boldsymbol{A}$ satisfies Equation (17), we say that $\boldsymbol{A}$ is the **stochastic matrix** of a Markov chain.

Back to the Markov chain

- In a Markov chain, a question of common interest is:
  - Given $\boldsymbol{p}_0$ at the beginning, what is the probability that m steps/transitions later the Markov chain will be at each state j?
- We determine the probability that the system (or the random surfer) is in state j after 1 step (1 transition) by using the following reasoning:

$$p_1(j) = \sum_{i=1}^{n} A_{ij}(1)\, p_0(i) \qquad (18)$$

State transition

$$p_1(j) = \sum_{i=1}^{n} A_{ij}(1) p_0(i), \tag{18}$$

where $A_{ij}(1)$ is the probability of going from i to j after 1 transition, and $A_{ij}(1) = A_{ij}$. We can write it with a matrix:

$$\boldsymbol{p}_1 = \boldsymbol{A}^{\boldsymbol{T}} \boldsymbol{p}_{\boldsymbol{0}} \tag{19}$$

In general, the probability distribution after k steps/transitions is:

$$\boldsymbol{p}_k = \boldsymbol{A}^{\boldsymbol{T}} \boldsymbol{p}_{k-1} \tag{20}$$

Stationary probability distribution

- By a Theorem of the Markov chain,
  - a finite Markov chain defined by the **stochastic matrix *A*** has a unique **stationary probability distribution** if ***A*** is **irreducible** and **aperiodic**.
- The stationary probability distribution means that after a series of transitions $\boldsymbol{p}_k$ will converge to a steady-state probability vector $\boldsymbol{\pi}$ regardless of the choice of the initial probability vector $\boldsymbol{p}_0$, i.e.,

$$\lim_{k \to \infty} \boldsymbol{p}_k = \boldsymbol{\pi} \qquad (21)$$

PageRank again

- When we reach the steady-state, we have $\boldsymbol{p}_k = \boldsymbol{p}_{k+1} = \pi$, and thus

$$\pi = \boldsymbol{A}^T \pi.$$

- π is the **principal eigenvector** of $\boldsymbol{A}^T$ with **eigenvalue** of 1.
- In PageRank, π is used as the PageRank vector $\boldsymbol{P}$. We again obtain Equation (15), which is re-produced here as Equation (22):

$$\boldsymbol{P} = \boldsymbol{A}^T \boldsymbol{P} \tag{22}$$

Is $P = \pi$ justified?

- Using the stationary probability distribution π as the PageRank vector is reasonable and quite intuitive because
  - it reflects the long-run probabilities that a random surfer will visit the pages.
  - A page has a high prestige if the probability of visiting it is high.

Back to the Web graph

- Now let us come back to the real Web context and see whether the above conditions are satisfied, i.e.,
  - whether ***A*** is a stochastic matrix and
  - whether it is irreducible and aperiodic.
- None of them is satisfied.
- Hence, we need to extend the ideal-case Equation (22) to produce the "actual PageRank" model.

A is a not stochastic matrix

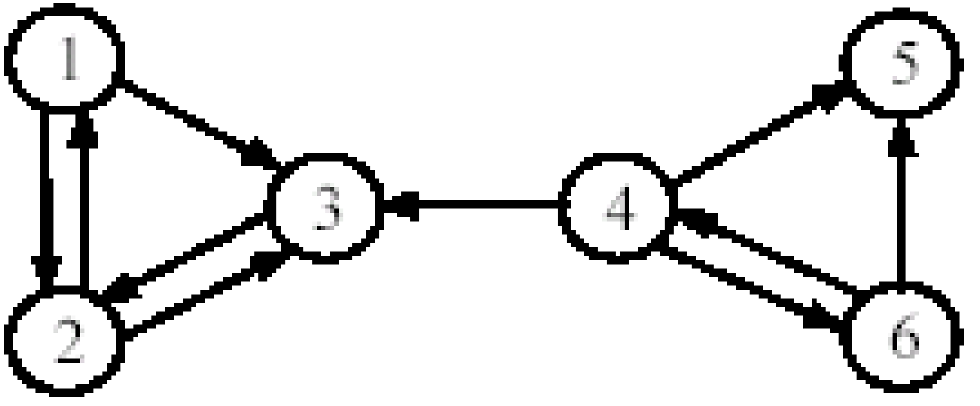
- ***A*** is the transition matrix of the Web graph

$$A_{ij} = \begin{cases} \dfrac{1}{O_i} & if\ (i,j) \in E \\ 0 & otherwise \end{cases}$$

- It does not satisfy equation (17) $\quad \sum_{j=1}^{n} A_{ij} = 1$

- because many Web pages have no out-links, which are reflected in transition matrix ***A*** by some rows of complete 0's.
  - Such pages are called the **dangling pages** (nodes).

An example Web hyperlink graph



$$\boldsymbol{A} = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

Fix the problem: two possible ways

1. Remove those pages with no out-links during the PageRank computation as these pages do not affect the ranking of any other page directly.
2. Add a complete set of outgoing links from each such page *i* to all the pages on the Web.

Let us use the second way

$$\overline{\boldsymbol{A}} = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

A is a not irreducible

- Irreducible means that the Web graph G is strongly connected.

Definition: A directed graph $G = (V, E)$ is strongly connected if and only if, for each pair of nodes $u, v \in V$, there is a path from u to v.

- A general Web graph represented by $\boldsymbol{A}$ is not irreducible because
  - for some pair of nodes u and v, there is no path from u to v.
  - In our example, there is no directed path from nodes 3 to 4.

A is a not aperiodic

- A state *i* in a Markov chain being periodic means that there exists a directed cycle that the chain has to traverse.

Definition: A state *i* is **periodic** with period $k > 1$ if k is the smallest number such that all paths leading from state *i* back to state *i* have a length that is a multiple of k.

  - If a state is not periodic (i.e., $k = 1$), it is **aperiodic**.
  - A Markov chain is **aperiodic** if all states are aperiodic.

An example: periodic

- Fig. 5 shows a periodic Markov chain with k = 3. Eg, if we begin from state 1, to come back to state 1 the only path is 1-2-3-1 for some number of times, say h. Thus any return to state 1 will take 3h transitions.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

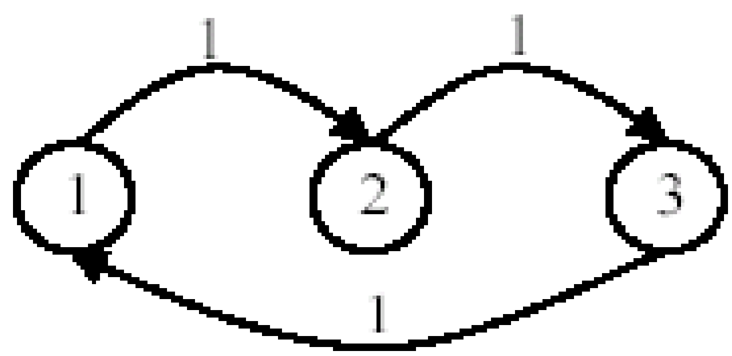


Fig. 5. A Periodic Markov chain with $k = 3$.

Deal with irreducible and aperiodic

- It is easy to deal with the above two problems with a single strategy.
- Add a link from each page to every page and give each link a small transition probability controlled by a parameter *d*.
- Obviously, the augmented transition matrix becomes **irreducible** and **aperiodic**

Improved PageRank

- After this augmentation, at a page, the random surfer has two options
  - With probability *d*, he randomly chooses an out-link to follow.
  - With probability 1-*d*, he jumps to a random page
- Equation (25) gives the improved model,

$$\boldsymbol{P} = ((1-d)\frac{\boldsymbol{E}}{n} + d\boldsymbol{A}^T)\boldsymbol{P} \qquad (25)$$

where $\boldsymbol{E}$ is $\boldsymbol{ee}^T$ ($\boldsymbol{e}$ is a column vector of all 1's) and thus $\boldsymbol{E}$ is a $n \times n$ square matrix of all 1's.

Follow our example

$$(1-d)\frac{\boldsymbol{E}}{n}+d\boldsymbol{A}^T=\begin{pmatrix} 1/60 & 7/15 & 1/60 & 1/60 & 1/6 & 1/60 \\ 7/15 & 1/60 & 11/12 & 1/60 & 1/6 & 1/60 \\ 7/15 & 7/15 & 1/60 & 19/60 & 1/6 & 1/60 \\ 1/60 & 1/60 & 1/60 & 1/60 & 1/6 & 7/15 \\ 1/60 & 1/60 & 1/60 & 19/60 & 1/6 & 7/15 \\ 1/60 & 1/60 & 1/60 & 19/60 & 1/6 & 1/60 \end{pmatrix}$$

The final PageRank algorithm

- $(1-d)\boldsymbol{E}/n + d\boldsymbol{A}^T$ is a **stochastic matrix** (transposed). It is also **irreducible** and **aperiodic**
- If we scale Equation (25) so that $\boldsymbol{e}^T\boldsymbol{P} = n$,

$$\boldsymbol{P} = (1-d)\boldsymbol{e} + d\boldsymbol{A}^T\boldsymbol{P} \qquad (27)$$

- PageRank for each page *i* is

$$P(i) = (1-d) + d\sum_{j=1}^{n} A_{ji}P(j) \qquad (28)$$

The final PageRank (cont …)

- (28) is equivalent to the formula given in the PageRank paper

$$P(i) = (1 - d) + d \sum_{(j,i) \in E} \frac{P(j)}{O_j}$$

- The parameter *d* is called the **damping factor** which can be set to between 0 and 1. *d* = 0.85 was used in the PageRank paper.

Compute PageRank

- Use the power iteration method

PageRank-Iterate(G)

$$
\begin{aligned}
& \boldsymbol{P}_0 \leftarrow \boldsymbol{e}/n \\
& k = 1 \\
& \textbf{repeat} \\
& \qquad \boldsymbol{P}_{k+1} \leftarrow (1-d)\boldsymbol{e} + d\boldsymbol{A}^T \boldsymbol{P}_k; \\
& \qquad k = k + 1; \\
& \textbf{until } \|\boldsymbol{P}_{k+1} - \boldsymbol{P}_k\|_1 < \varepsilon \\
& \text{return } \boldsymbol{P}_{k+1}
\end{aligned}
$$

Fig. 6. The power iteration method for PageRank

Advantages of PageRank

- **Fighting spam**. A page is important if the pages pointing to it are important.
  - Since it is not easy for Web page owner to add in-links into his/her page from other important pages, it is thus not easy to influence PageRank.
- **PageRank is a global measure and is query independent**.
  - PageRank values of all the pages are computed and saved off-line rather than at the query time.
- Criticism: Query-independence. It could not distinguish between pages that are authoritative in general and pages that are authoritative on the query topic.

Road map

- **PageRank**
- **HITS**
- **Summary**

HITS

- HITS stands for **Hypertext Induced Topic Search**.
- Unlike PageRank which is a static ranking algorithm, HITS is search query dependent.
- When the user issues a search query,
  - HITS first expands the list of relevant pages returned by a search engine and
  - then produces two rankings of the expanded set of pages, **authority ranking** and **hub ranking**.

Authorities and Hubs

Authority: Roughly, a authority is a page with many in-links.

- The idea is that the page may have good or authoritative content on some topic and
- thus many people trust it and link to it.

Hub: A hub is a page with many out-links.

- The page serves as an organizer of the information on a particular topic and
- points to many good authority pages on the topic.

Examples

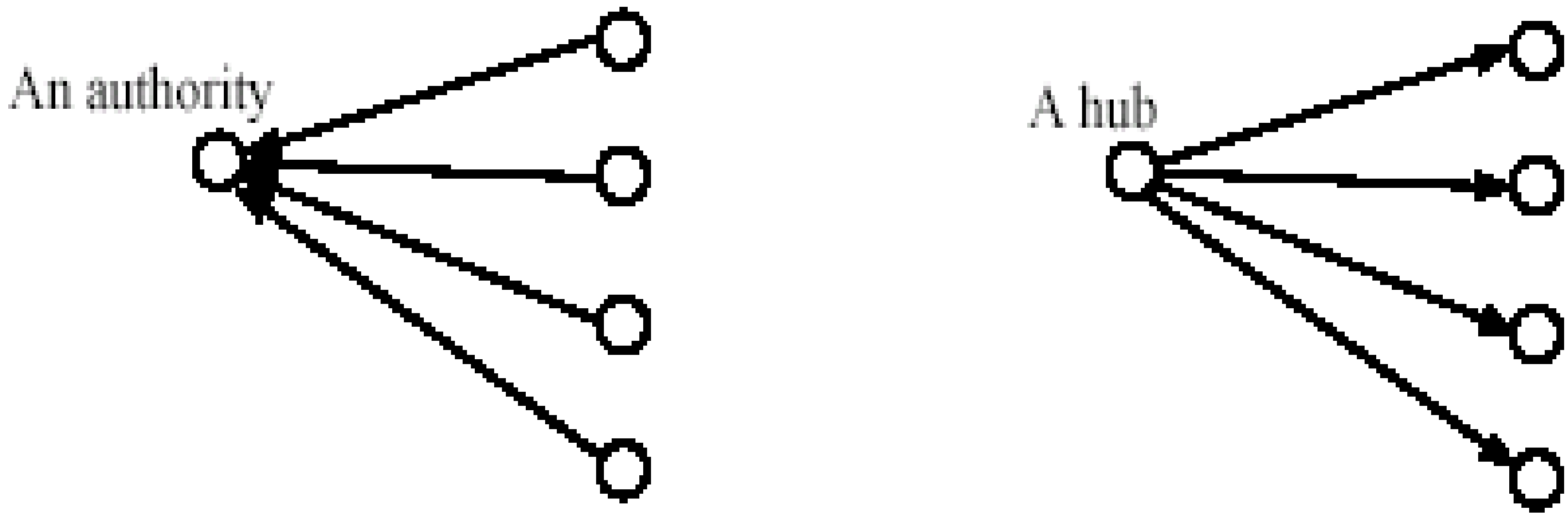


Fig. 7. An authority page and a hub page

The key idea of HITS

- A good hub points to many good authorities, and
- A good authority is pointed to by many good hubs.
- Authorities and hubs have a **mutual reinforcement relationship**. Fig. 8 shows some densely linked authorities and hubs (a **bipartite sub-graph**).

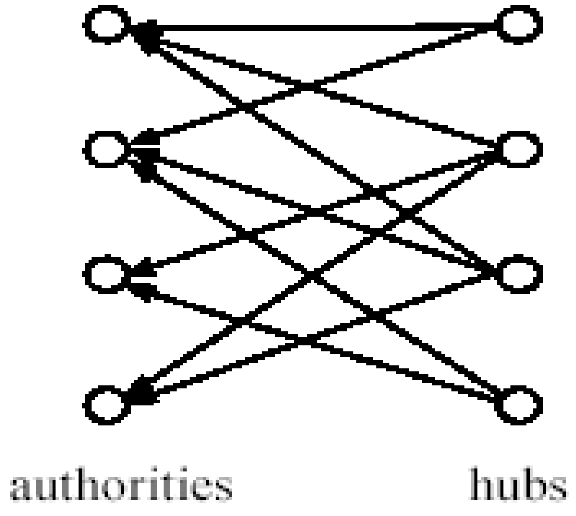


Fig. 8. A densely linked set of authorities and hubs

The HITS algorithm: Grab pages

- Given a broad search query, *q*, HITS collects a set of pages as follows:
  - It sends the query *q* to a search engine.
  - It then collects *t* (*t* = 200 is used in the HITS paper) highest ranked pages. This set is called the **root** set *W*.
  - It then grows *W* by including any page pointed to by a page in *W* and any page that points to a page in *W*. This gives a larger set *S*, **base set**.

The link graph G

- HITS works on the pages in *S*, and assigns every page in *S* an **authority score** and a **hub score**.
- Let the number of pages in *S* be *n*.
- We again use *G* = (*V*, *E*) to denote the hyperlink graph of *S*.
- We use ***L*** to denote the adjacency matrix of the graph.

$$L_{ij} = \begin{cases} 1 & if\,(i, j) \in E \\ 0 & otherwise \end{cases}$$

The HITS algorithm

- Let the authority score of the page *i* be *a*(*i*), and the hub score of page *i* be *h*(*i*).
- The mutual reinforcing relationship of the two scores is represented as follows:

$$a(i) = \sum_{(j,i)\in E} h(j) \qquad (31)$$

$$h(i) = \sum_{(i,j)\in E} a(j) \qquad (32)$$

HITS in matrix form

- We use $\boldsymbol{a}$ to denote the column vector with all the authority scores,

  $\boldsymbol{a} = (a(1), a(2), \ldots, a(n))^T$, and
- use $\boldsymbol{h}$ to denote the column vector with all the authority scores,

  $\boldsymbol{h} = (h(1), h(2), \ldots, h(n))^T$,
- Then,

$$\boldsymbol{a} = \boldsymbol{L}^T\boldsymbol{h} \qquad (33)$$

$$\boldsymbol{h} = \boldsymbol{L}\boldsymbol{a} \qquad (34)$$

Computation of HITS

- The computation of authority scores and hub scores is the same as the computation of the PageRank scores, using power iteration.
- If we use $\boldsymbol{a}_k$ and $\boldsymbol{h}_k$ to denote authority and hub vectors at the kth iteration, the iterations for generating the final solutions are

$$a_k = L^T L a_{k\text{-}1} \tag{35}$$

$$h_k = L L^T h_{k\text{-}1} \tag{36}$$

starting with

$$a_0 = h_0 = (1, 1, \ldots, 1), \tag{37}$$

The algorithm

HITS-Iterate(G)

$a_0 = h_0 = (1, 1, \ldots, 1)$;
$k = 1$
Repeat

$a_k = L^T L a_{k-1}$;

$h_k = L L^T h_{k-1}$;

normalize a_k;
normalize h_k;
$k = k + 1$;

until a_k and h_k do not change significantly;
return a_k and h_k

. **Fig. 9.** The HITS algorithm based on power iteration

Relationships with co-citation and bibliographic coupling

- Recall that co-citation of pages *i* and *j*, denoted by C_{ij}, is

$$C_{ij} = \sum_{k=1}^{n} L_{ki} L_{kj} = (\boldsymbol{L}^T \boldsymbol{L})_{ij}$$

  - the authority matrix ($\boldsymbol{L}^T\boldsymbol{L}$) of HITS is the co-citation matrix ***C***
- bibliographic coupling of two pages *i* and *j*, denoted by B_{ij} is

$$B_{ij} = \sum_{k=1}^{n} L_{ik} L_{jk} = (\boldsymbol{L}\boldsymbol{L}^T)_{ij},$$

  - the hub matrix ($\boldsymbol{L}\boldsymbol{L}^T$) of HITS is the bibliographic coupling matrix ***B***

Strengths and weaknesses of HITS

- Strength: its ability to rank pages according to the query topic, which may be able to provide more relevant authority and hub pages.
- Weaknesses:
  - It is easily spammed. It is in fact quite easy to influence HITS since adding out-links in one's own page is so easy.
  - Topic drift. Many pages in the expanded set may not be on topic.
  - Inefficiency at query time: The query time evaluation is slow. Collecting the root set, expanding it and performing eigenvector computation are all expensive operations

Road map

- PageRank
- HITS
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Summary

- In Link Analysis chapter, we introduced
  - Social network analysis, centrality and prestige
  - Co-citation and bibliographic coupling
  - PageRank, which powers Google
  - HITS
- Yahoo! and MSN have their own link-based algorithms as well, but not published.
- **Important to note**: Hyperlink based ranking is not the only algorithm used in search engines. In fact, it is combined with many content based factors to produce the final ranking presented to the user.

Summary

- Links can also be used to find communities, which are groups of content-creators or people sharing some common interests.
  - Web communities
  - Email communities
  - Named entity communities
- Focused crawling: combining contents and links to crawl Web pages of a specific topic.
  - Follow links and
  - Use learning/classification to determine whether a page is on topic.